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Comparison of Analytical and Superposition Solutions of the Transient Liquid Crystal Technique

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The transient liquid crystal technique has been widely used in heat transfer measurement and, due to the nature of the transient test, the mainstream temperature changes over time. In many previous studies, the change of mainstream temperature was assumed as a series of step changes and Duhamel's superposition theorem was applied to evaluate the heat transfer coefficient. In this paper, the error in calculating the heat transfer coefficient by the series-of-step-changes assumption with Duhamel's superposition theorem is discussed and the analytical solution for the curve-fitted mainstream temperature with *n*th-order polynomials is presented. The solution using the series of step changes showed a high dependency on the size of the time step, thus on the sampling ratio of the mainstream temperature measurement, and higher error was induced for the higher heat transfer coefficient. It is recommended that the size of time step in the series of step changes should be minimized and that the use of an analytical solution would be a better choice in the transient liquid crystal test. It is expected that the presented analytical solution for the curve-fitted mainstream temperature with polynomials could be applied to many slow transient liquid crystal tests.

Nomenclature

a = coefficient for polynomial least-square regression
c = thermal capacity of the test section
h = heat transfer coefficient
k = thermal conductivity of the test section
m = slope of the ramp
N = order of polynomial $+1$
n = order of polynomial
T = temperature
t = color change time
x = distance from the test surface
α = thermal diffusivity of the test section
β = $h\sqrt{t}/\sqrt{\rho c k}$
β_p = $h\sqrt{\alpha}/k$
β_τ = $h\sqrt{\tau}/\sqrt{\rho c k}$
Γ = gamma function
θ = $T - T_i$
ρ = density of the test section
τ = time constant

Subscripts

aw = adiabatic wall
i = initial state
m = mainstream
w = surface
∞ = value as $t \to \infty$

I. Introduction

THE transient liquid crystal technique has been widely used in heat transfer measurement because the technique has high spatial resolution, requires relatively cheap equipment, and can be applied to complex geometries. The applications of the transient liquid crystal technique were reviewed by Baughn [1], Ekkad and Han [2], Ireland et al. [3], and Ireland and Jones [4]. In the transient liquid crystal technique, the test surface is assumed to be a one-dimensional semi-infinite solid with a convective boundary condition, and a sudden change in the mainstream temperature (or the mainstream itself) is applied. As the surface, coated in liquid crystals, is heated or cooled by the mainstream, the change of color or intensity of the liquid crystals is recorded and the heat transfer coefficient is calculated by using the transient time between the initial temperature and another temperature, specified in advance. The simplest case can be obtained if the mainstream temperature shows a step change. However, in many tests, the step change in the mainstream temperature cannot be achieved and the mainstream temperature varies during the transient test. By many researchers, the change in mainstream temperature was assumed as a series of step changes, and Duhamel's superposition theorem was applied to calculate the heat transfer coefficient. In this paper, it will be shown that the assumption of a series of step changes could induce a number of errors in the calculated heat transfer coefficient and, consequently, the analytical solution for slow transient liquid crystal tests will be presented.

II. Solutions for the Transient Liquid Crystal Technique

In the transient liquid crystal technique, the test surface is assumed to be a one-dimensional semi-infinite solid model with a convective boundary condition. The one-dimensional conduction equation and the initial and boundary conditions for the transient liquid crystal test are as follows:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

$$\text{at } t = 0, \qquad T = T_i \tag{2}$$

$$\text{at } x = 0, \qquad -k\frac{\partial T}{\partial x} = h(T_m - T) \tag{3}$$

$$\text{as } x \to \infty, \qquad T = T_i \tag{4}$$

Depending on the test facility, the mainstream temperature can be modeled as a step change, a series of step changes, an exponential function, a series of exponential functions, a ramp, or a series of

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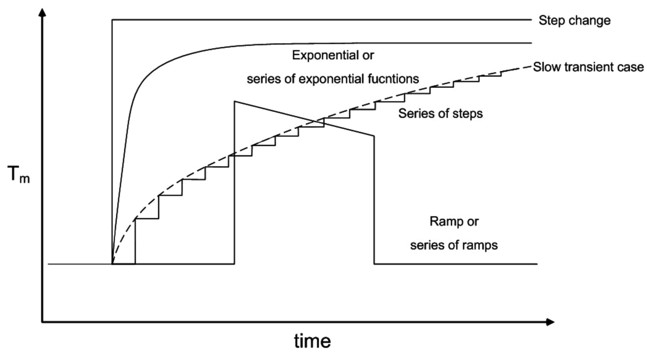


Fig. 1 Various mainstream temperature models.

ramps, as shown in Fig. 1. Researchers have derived solutions for Eqs. (1–4) for different mainstream temperature models.

A. Step Change or Series of Step Changes

If the mainstream temperature shows a step change, the temperature at the test surface can be expressed as [5]

$$T_w = T_i + (T_m - T_i)\left[1 - \exp\left(\frac{h^2 t}{\rho c k}\right) erfc\left(\frac{h\sqrt{t}}{(\rho c k)^{1/2}}\right)\right] \tag{5}$$

In a real test with a conventional air heater, it is very difficult or even impossible to get a pure step change in the mainstream temperature. Many researchers have used a series of step changes with Duhamel's superposition theorem. In this case, the temperature at $x = 0$ can be written as Eq. (6) [6]:

$$T_w = T_i + \sum_{i=1}^{n}(T_{m_i} - T_{m_{i-1}}) \times \left[1 - \exp\left(\frac{h^2(t - t_i)}{\rho c k}\right) erfc\left(\frac{h\sqrt{t - t_i}}{(\rho c k)^{1/2}}\right)\right] \tag{6}$$

B. Exponential Function or Series of Exponential Functions

Newton et al. [7] used a mesh heater to heat mainstream air and they modeled the mainstream temperature variation using an exponential function or a series of exponential functions. If the mainstream temperature can be expressed with a single exponential function, the temperature at the test surface is expressed as

$$\frac{T_w - T_i}{T_{\mathrm{aw},\infty} - T_i} = 1 - \frac{1}{1 + \beta_\tau^2}\exp(\beta^2) \times erfc(\beta) - e^{-t/\tau}\frac{\beta_\tau^2}{1 + \beta_\tau^2}\left\{1 + \frac{1}{\beta_\tau}\left[\frac{1}{\pi}\sqrt{\frac{t}{\tau}} + \frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{n}e^{-n^2/4}\sinh\left(n\sqrt{\frac{t}{\tau}}\right)\right]\right\} \tag{7}$$

where

$$\beta = \frac{h\sqrt{t}}{\sqrt{\rho c k}}, \qquad \beta_\tau = \frac{h\sqrt{\tau}}{\sqrt{\rho c k}},$$
$$T_m(t) = T_{m,o} + (T_{m,\infty} - T_{m,o})(1 - e^{-t/\tau})$$

In the actual test, a single exponential function is not sufficient to model the mainstream temperature variations and, consequently, Newton et al. [7] used a series of exponential functions. In this case, the solution for Eqs. (1–4) is

$$\frac{T_w - T_i}{T_{\mathrm{aw},\infty} - T_i} = \sum_{j=1}^{m}\frac{T_{m,j}}{T_{\mathrm{aw},\infty} - T_i}\left\{1 - \frac{1}{1 + \beta_\tau^2}\exp(\beta^2) \times erfc(\beta) - e^{-t/\tau}\frac{\beta_\tau^2}{1 + \beta_\tau^2}\left\{1 + \frac{1}{\beta_\tau}\left[\frac{1}{\pi}\sqrt{\frac{t}{\tau}} + \frac{2}{\pi}\sum_{1}^{\infty}\frac{1}{n}e^{-n^2/4}\sinh\left(n\sqrt{\frac{t}{\tau}}\right)\right]\right\}\right\} \tag{8}$$

where

$$\beta = \frac{h\sqrt{t}}{\sqrt{\rho c k}}, \qquad \beta_\tau = \frac{h\sqrt{\tau}}{\sqrt{\rho c k}},$$
$$T_m(t) = T_{a,0} + \sum_{j=1}^{m} T_{m,j}(1 - e^{-t/\tau_j})$$

C. Ramp or Series of Ramps

Ireland and Jones [4] and Ling et al. [8] used battery-powered heaters, and their mainstream temperature was modeled with a ramp or a series of ramps. For a single-ramp case, the solution is

$$T_w = T_i + mt\left[1 - \frac{2}{\beta\sqrt{\pi}} + \frac{1 - \exp(\beta)^2 erfc(\beta)}{\beta^2}\right] \tag{9}$$

where

$$\beta = \frac{h\sqrt{t}}{\sqrt{\rho c k}}$$

For a series-of-ramps case, Ling et al. [8] presented that the solution for surface temperature is

$$T_w = T_i + T_{\mathrm{step}} + \sum_{i=1}^{n} m_i U(t - t_i) \tag{10}$$

where

$$T_{\mathrm{step}} = (T_{m,0} - T_i)[1 - \exp(\beta^2) erfc(\beta)],$$
$$\mathrm{U}(t - t_i) = m_i t\left(1 - \frac{2}{\beta i} + \frac{1 - \exp(\beta_i)^2 erfc(\beta_i)}{\beta_i^2}\right),$$
$$\beta = \frac{h\sqrt{t}}{\sqrt{\rho c k}}, \qquad \beta_i = \frac{h\sqrt{t - t_i}}{\sqrt{\rho c k}}$$

D. Curve-Fitting with *n*th-Order Polynomials

In many internal transient heat transfer tests with conventional air heaters, the mainstream temperature shows a slow transition, as shown in Fig. 1, and it is possible to model the variation of the mainstream temperature with nth-order polynomials. In this study, the change of the mainstream temperature is curve-fitted with an nth-order polynomial, as shown in Eq. (11):

$$\theta_m = T_m - T_i = \sum_{n=1}^{N} a_n \frac{t^{n-1}}{\Gamma(n)} \tag{11}$$

where $N = n + 1$, $\Gamma(n)$ is the gamma function, and a_n is the coefficient for the nth-order polynomial.

If a new parameter θ is defined as $\theta = T - T_i$, Eqs. (1–4) can be written as

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{12}$$

$$\theta = 0 \quad \text{for } x \to \infty \tag{13}$$

$$\theta = 0 \quad \text{for } t = 0 \tag{14}$$

$$-k\frac{\partial\theta}{\partial x} = h(\theta_m - \theta) \quad \text{at } x = 0 \tag{15}$$

The solution of the preceding equations at $x = 0$ can be expressed as

$$\begin{aligned}\theta_w &= T_w - T_i \\ &= \sum_{n=1}^{N} a_n \Bigg[\sum_{m=1}^{n} \frac{1}{\beta_p^{2(m-1)}} \frac{t^{n-m}}{(n-m)!} - \frac{1}{\beta_p^{2(n-1)}} e^{t\beta_p^2} erfc(\beta_p \sqrt{t}) \\ &\quad - \sum_{i=1}^{n-1} \frac{1}{\beta_p^{2i-1}} \frac{2^{n-i} t^{n-i-\frac{1}{2}}}{(1\cdot 3\cdot 5\cdot 7\cdots\cdots[2(n-i)-1])\sqrt{\pi}}\Bigg]\end{aligned} \tag{16}$$

where

$$\beta_p = \frac{h\sqrt{\alpha}}{k}$$

Equation (16) can be applied to any transient liquid crystal test as long as the mainstream temperature variation with time can be expressed with nth-order polynomials.

III. Error in the Heat Transfer Coefficient Calculation with the Series-of-Step-Changes Model

To compare the calculated heat transfer coefficients by analytical solution [Eq. (16)] and series of step changes with Duhamel's superposition theorem [Eq. (6)], three mainstream profiles were considered, as shown in Fig. 2, in which cases 1 and 3 present relatively fast and slow transient mainstream temperatures, respectively. In a transient test, the profile of the mainstream temperature depends on the type of heater or distance between the heater and the test section. Three different mainstream profiles were used to simulate different test conditions.

For the heat transfer coefficient calculation, the mainstream temperatures in Fig. 2 were curve-fitted with tenth-order polynomials. The least-squares regression with tenth-order polynomials for all cases gave the correlation coefficients r of more than 0.99999999, which means an almost perfect fit for given data [9]. In the calculation, the properties of the test section, the initial temperature, and the wall temperature at t were assumed as $\alpha = 0.1389 \times 10^{-6}$ m^2/s, $k = 0.2$ W/m $\cdot$ °C, $T_i = 28.0°$ C, and $T_w = 36.0°$ C, respectively.

Figure 3 presents the analytical solutions for each case with various color change times. For a given color change time, a higher mainstream temperature results in a lower heat transfer coefficient. Figure 3 also shows that the calculated heat transfer coefficients are very sensitive to time at shorter color change times. It is expected that if the color change time is relatively short (if the heat transfer coefficient is high), a small error in the color change time would cause a severe error in the calculated heat transfer coefficient. It was reported that the uncertainty at the high heat transfer coefficient is high due to other sources of measurement uncertainties [10]. Thus, for the high heat transfer coefficient calculated by Duhamel's superposition theorem, the calculated heat transfer coefficient would include large uncertainty by both experimental uncertainty and the error caused by the series-of-step-changes assumption, which will be presented in a later part of this paper.

During typical transient liquid crystal tests, the change of the mainstream temperature is measured by thermocouples at the limited sampling rate. If the variation of mainstream temperature is assumed as a series of step changes, the measured mainstream temperatures are used in evaluating the heat transfer coefficient by Eq. (6). At this point, it will be very important to check that the sampling rate of the mainstream temperature measuring device is enough to calculate an accurate heat transfer coefficient.

Figures 4–6 present the relative error of calculated heat transfer coefficients by Eq. (6) for the cases 1, 2, and 3, respectively. The relative error is defined as

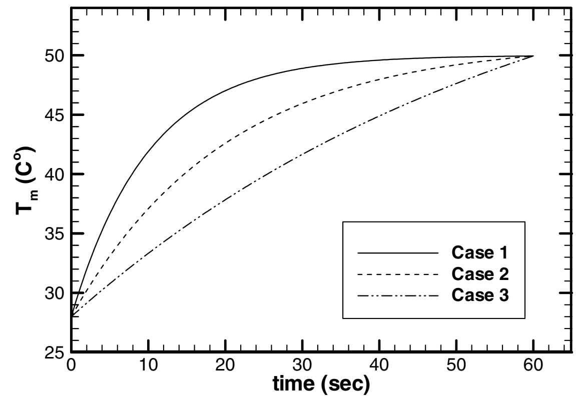


Fig. 2 Example mainstream temperature profiles.

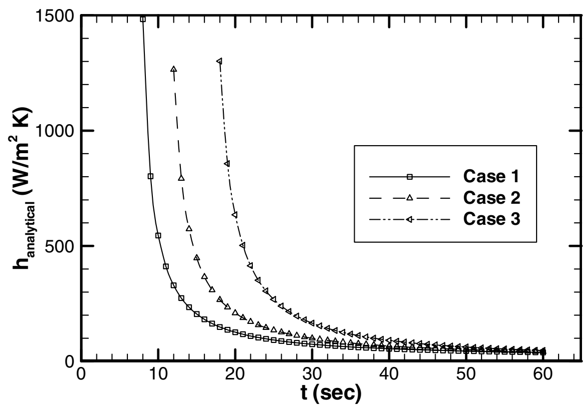


Fig. 3 Analytical solutions for various color change time.

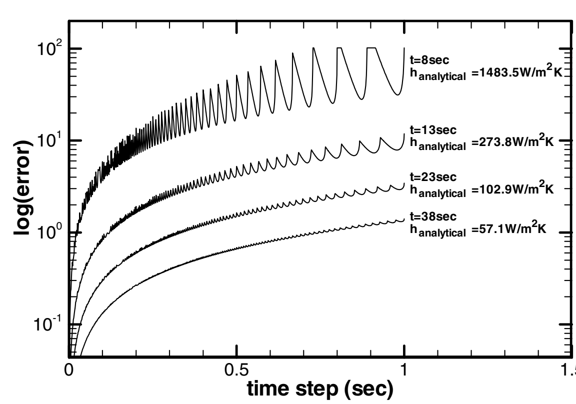


Fig. 4 Relative error on heat transfer coefficient calculated by Eq. (6): Case 1.

$$\text{error} = \left|\frac{h_{\text{analytical}} - h_{\text{superposition}}}{h_{\text{analytical}}}\right| \% \tag{17}$$

In Figs. 4–6, all cases show that the relative errors increase as the size of the time step increases. The larger time step means a lower sampling rate in the mainstream temperature measurement. The increase in the relative error is more severe for the higher heat transfer case. For the highest heat transfer coefficient for each case, the relative errors are close to 100%. With the size of the time step at 1 s, the relative error for the lowest heat transfer coefficient is still about 1%. The relative error would increase further with a greater increase in the size of the time step. Many researchers have used the series of step changes with Duhamel's superposition theorem in the transient liquid crystal tests and no attention was paid to the time step

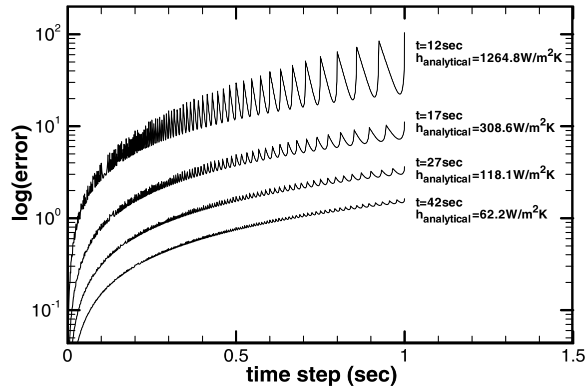


Fig. 5 Relative error on heat transfer coefficient calculated by Eq. (6): Case 2.

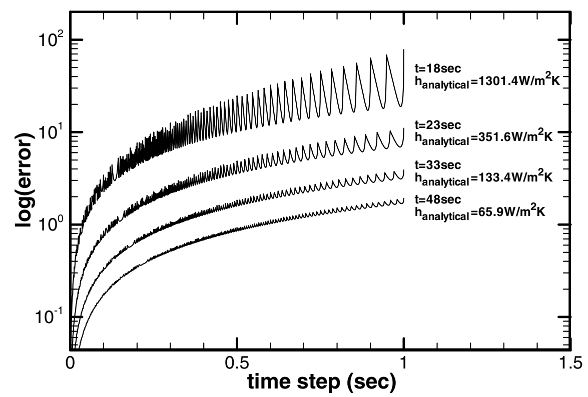
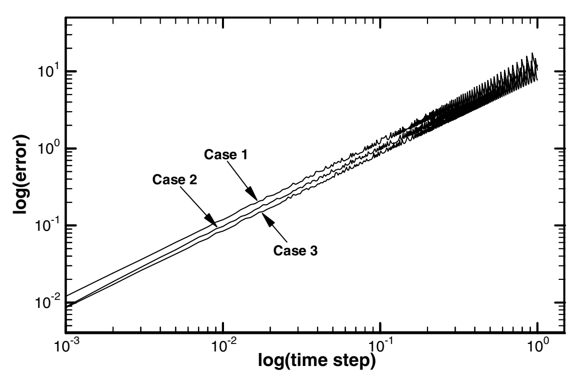
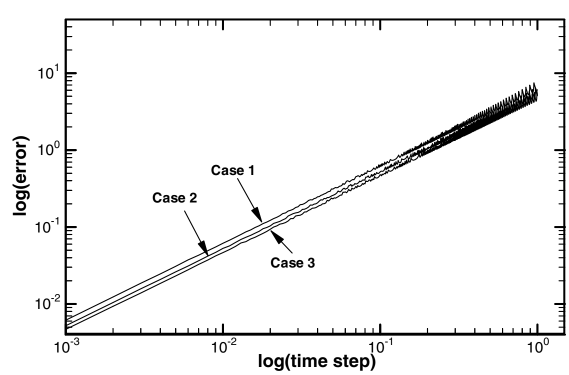


Fig. 6 Relative error on heat transfer coefficient calculated by Eq. (6): Case 3.



a) h=400 W/m^2 K



b) h=200 W/m^2 K

Fig. 7 Relative error for same heat transfer coefficient calculated by Eq. (6).

(sampling rate of mainstream temperature) in the heat transfer coefficient calculation. Figures 4–6 clearly show that the size of the time step should be minimized (the sampling rate in the mainstream temperature measurement should be increased) for an accurate heat transfer coefficient calculation by Eq. (6).

The error shown in Figs. 4–6 does not include any experimental uncertainty caused by liquid crystal calibration, lighting conditions, the angle of the camera and light, temperature ratios, 2D conduction effects, or any other sources of measurement uncertainty. Thus, when the series of step changes is applied to the transient liquid crystal technique, the size of the time step in the calculation should be minimized. Otherwise, unwanted error can be included in the calculated heat transfer coefficient, in addition to the measurement uncertainties.

Figure 7 shows the relative error for the same heat transfer coefficient and different mainstream temperature profile. For all the mainstream temperature cases, the case with the higher heat transfer coefficient shows the highest relative error. For both heat transfer coefficient cases, the relative error for case 1 is higher than those of the other cases. Because the slope on the mainstream temperature for case 1 is higher than those for other cases, the error in the assumption of series of step changes is higher for case 1. A larger error in the series-of-step-changes assumption causes larger relative errors. Thus, if the series-of-step-changes assumption is applied to the mainstream temperature with a larger slope, the size of the time step should be further decreased.

Because the measurement uncertainties in the typical transient liquid crystal test is about 10%, the error induced by using the series-of-steps model with Duhamel's superposition theorem cannot be disregarded. It should be emphasized that caution should be paid in applying Duhamel's superposition theorem to the transient liquid crystal technique. Also, the analytical solutions would make a better choice in calculating the heat transfer coefficient for the transient liquid crystal test. The heat transfer coefficient by analytical solution will prevent unwanted errors in the calculation, and the author's experience shows that the required CPU time is reduced by at least half.

IV. Application of the Analytical Solution to a Slow Transient Test

The analytical solution for slow transient tests was applied to the heat transfer measurement in a rectangular duct with perforated blockage walls. Figure 8 shows the schematic of the test facility. The test facility consisted of a blower, an electric heater, a Venturi flow meter, two pneumatic valves, and a test section with three blockage walls. Each blockage wall was equipped with seven staggered holes. Figure 9 presents the detailed geometry of the heat transfer measurement surface.

The test section was made of 10-mm-thick transparent polycarbonate, and seven holes with a diameter of 10.3 mm were fabricated on each blockage wall. Heated air was bypassed until the air temperature reached a preset value and subsequently diverted to the test section by pneumatic valves after the air temperature reached the predetermined value. For the mainstream temperature measurement, four T-type thermocouples were installed on each measurement plane, and the averaged temperature was used in the heat transfer coefficient calculation for each measurement plane. Figure 10 shows the measured and curve-fitted mainstream

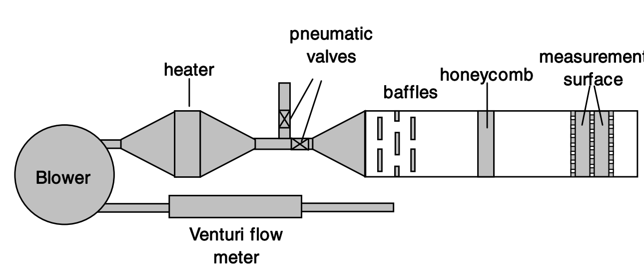


Fig. 8 Schematic of the test facility (not to scale).

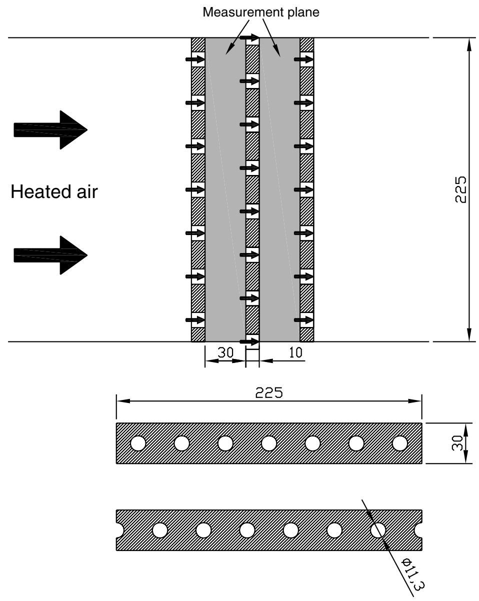


Fig. 9 Detailed view of heat transfer measurement surface.

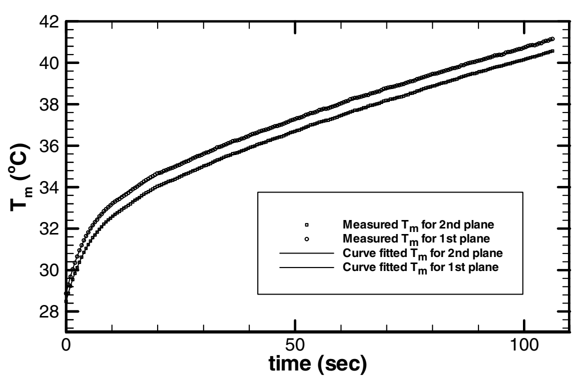


Fig. 10 Mainstream temperature for each measurement surface.

temperatures for each measurement plane. Both mainstream temperatures were curve-fitted with tenth-order polynomials, and the correlation coefficient r was more than 0.99999.

Black paint and liquid crystals (35C1W, Hallcrest) with a bandwidth of 1°C were sprayed on the heat transfer measurement planes, and a digital CCD camera and incandescent lamps were installed above the test section. During the heat transfer measurement, the color change of the liquid crystals was stored in a computer through an IEEE1394 cable at a rate of 30 frames per second in a digital-video-format Audio Video Interleave (AVI) file. A Matlab-based image processing program developed by the author was used to calculate the hue change time from the initial state to the reference hue value. From every pixel of each frame of the AVI file, RGB (red, green, and blue) values were calculated and converted to an 8-bit scale hue. The reference hue value was set as 50 and the elapsed time from test start to the reference hue value was calculated for every pixel.

For the same mainstream temperature and color-change-time matrix, heat transfer coefficients were calculated by Eqs. (6) and (16) and the relative error was evaluated. Figure 11 shows the distribution of heat transfer coefficient calculated by an analytical solution [Eq. (16)]. The results show that the jet impingement on the downstream wall increases the heat transfer, and the overall heat transfer coefficient on the second measurement plane is higher than that on the first measurement plane.

Figure 12 is the distribution of the relative error between the analytical solution and the results by the series-of-step-changes assumption with Duhamel's superposition theorem. Figure 12

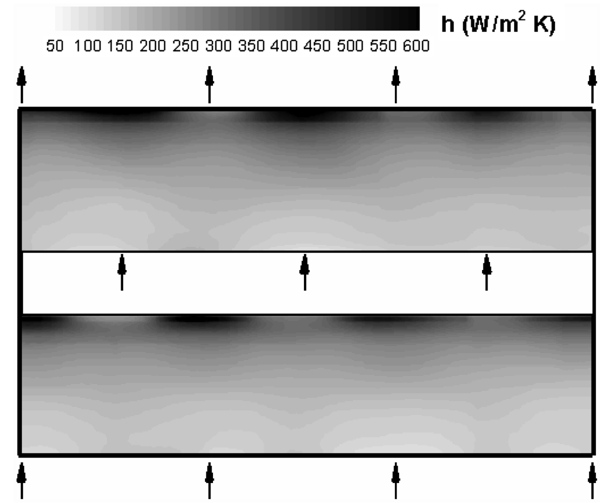
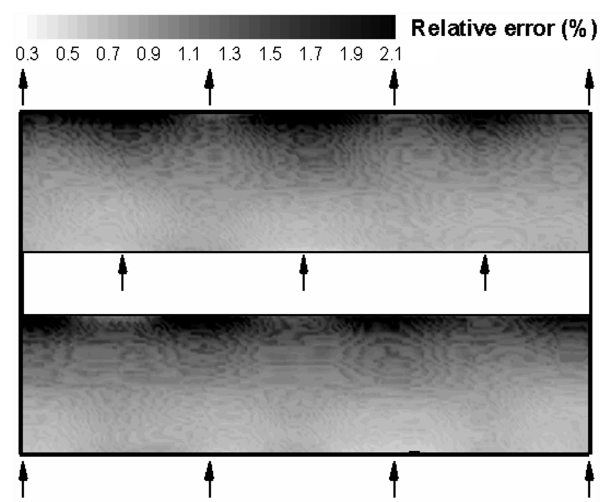
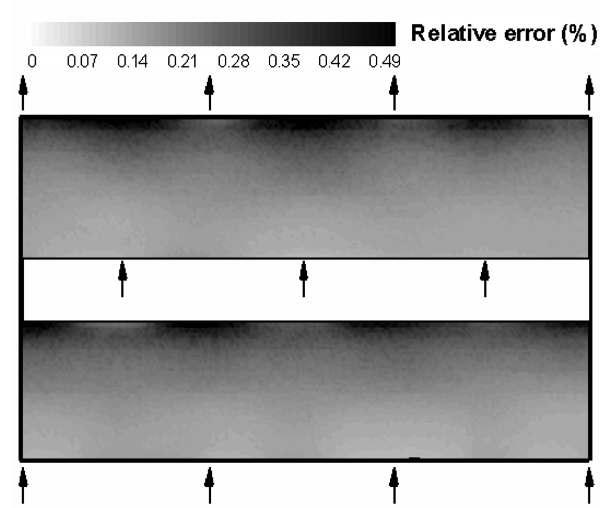


Fig. 11 Heat transfer coefficient calculated by Eq. (16).



a) Time step = 0.5 s



b) Time step = 0.1 s

Fig. 12 Relative error between results by Eqs. (6) and (16).

clearly shows that the smaller time step gives a lower relative error. For the time step of 0.5 s, the highest relative error is as high as 3.5%; however, the relative error for the time step of 0.1 s is less than 1%. For both time-step cases, higher relative errors occur at the points of high heat transfer coefficient. For the higher heat transfer coefficient, the relative error occurred by the series-of-steps assumption in the mainstream temperature would be higher than in the presented case.

As mentioned already, the relative error shown in Fig. 12 is not related to any measurement uncertainties. To avoid additional errors in the heat transfer coefficient calculation in the transient liquid crystal technique, the size of the time step in Eq. (6) should be minimized or the analytical solution should be applied.

V. Conclusions

The series-of-step-change assumptions on the mainstream temperature have been widely used in the transient liquid crystal technique. In this paper, the analytical solution for the slow transient liquid crystal test was presented and the analytical solution and the series of step changes with Duhamel's superposition theorem were compared for typical slow transient mainstream temperature profiles. Results showed that the accuracy of the solution by the series of step changes showed a high dependency on the size of the time step in the calculation, and a smaller size of the time step resulted in lower relative errors. The relative error is higher for the higher heat transfer coefficient and higher slope in the mainstream temperature. Based on the results shown in this paper, it is concluded that the time step in the series of step changes with Duhamel's superposition theorem should be minimized to reduce calculation errors. The analytical solution would be a better choice in the heat transfer coefficient calculation, because the analytical solution guarantees accurate heat transfer coefficient and reduces the computation time by at least half. It is expected that the analytical solution presented in this paper can be applied in any slow transient test as long as the mainstream temperature can be expressed in the nth-order polynomial.

Acknowledgments

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